

# On the $f$ -Norm Ergodicity of Markov Processes in Continuous Time

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## Abstract

Consider a Markov process  $\Phi = \{\Phi(t) : t \geq 0\}$  evolving on a Polish space  $\mathsf{X}$ . A version of the  $f$ -Norm Ergodic Theorem is obtained: Suppose that the process is  $\psi$ -irreducible and aperiodic. For a given function  $f: \mathsf{X} \rightarrow [1, \infty)$ , under suitable conditions on the process the following are equivalent:

- (i) There is a unique invariant probability measure  $\pi$  satisfying  $\int f d\pi < \infty$ .
- (ii) There is a closed set  $C$  satisfying  $\psi(C) > 0$  that is “self  $f$ -regular.”
- (iii) There is a function  $V: \mathsf{X} \rightarrow (0, \infty]$  that is finite on at least one point in  $\mathsf{X}$ , for which the following Lyapunov drift condition is satisfied,

$$\mathcal{D}V \leq -f + b\mathbb{I}_C, \quad (\text{V3})$$

where  $C$  is a closed small set and  $\mathcal{D}$  is the extended generator of the process.

For discrete-time chains the result is well-known. Moreover, in that case, the ergodicity of  $\Phi$  under a suitable norm is also obtained: For each initial condition  $x \in \mathsf{X}$  satisfying  $V(x) < \infty$ , and any function  $g: \mathsf{X} \rightarrow \mathbb{R}$  for which  $|g|$  is bounded by  $f$ ,

$$\lim_{t \rightarrow \infty} \mathbf{E}_x[g(\Phi(t))] = \int g d\pi.$$

Possible approaches are explored for establishing appropriate versions of corresponding results in continuous time, under appropriate assumptions on the process  $\Phi$  or on the function  $g$ .

**Keywords:** Markov process, continuous time, generator, stochastic Lyapunov function, ergodicity

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## 1 Introduction

Consider a Markov process  $\Phi = \{\Phi(t) : t \geq 0\}$  in continuous time, evolving on a Polish space  $\mathsf{X}$ , equipped with its Borel  $\sigma$ -field  $\mathcal{B}$ . Assume it is a nonexplosive Borel right process: It satisfies the strong Markov property and has right-continuous sample paths [1, 9].

The distribution of the process  $\Phi$  is described by the initial condition  $\Phi(0) = x \in \mathsf{X}$  and the transition semigroup: For any  $t \geq 0$ ,  $x \in \mathsf{X}$ ,  $A \in \mathcal{B}$ ,

$$P^t(x, A) := \mathbb{P}_x\{\Phi(t) \in A\} := \Pr\{\Phi(t) \in A \mid \Phi(0) = x\}.$$

A set  $C$  is called *small* if there is probability measure  $\nu$  on  $(\mathsf{X}, \mathcal{B})$ , a time  $T > 0$ , and a constant  $\varepsilon > 0$  such that,

$$P^T(x, A) \geq \varepsilon \nu(A), \quad \text{for every } A \in \mathcal{B}.$$

It is assumed that the process is  $\psi$ -irreducible and aperiodic, where  $\psi$  is a probability measure on  $(\mathsf{X}, \mathcal{B})$ . This means that for each set  $A \in \mathcal{B}$  satisfying  $\psi(A) > 0$ , and each  $x \in \mathsf{X}$ ,

$$P^t(x, A) > 0, \quad \text{for all } t \text{ sufficiently large.}$$

It follows that there is a countable covering of the state space by small sets [8, Prop. 3.4].

The Lyapunov theory considered in this paper and in our previous work [5, 9] is based on the *extended generator* of  $\Phi$ , denoted  $\mathcal{D}$ . A function  $h : \mathsf{X} \rightarrow \mathbb{R}$  is in the domain of  $\mathcal{D}$  if there exists a function  $g : \mathsf{X} \rightarrow \mathbb{R}$  such that the stochastic process defined by,

$$M(t) = h(\Phi(t)) - \int_0^t g(\Phi(s)) ds, \quad t \geq 0, \quad (1)$$

is a *local martingale*, for each initial condition  $\Phi(0)$  [1, 13]. We then write  $g = \mathcal{D}h$ .

For example, consider a diffusion on  $\mathsf{X} = \mathbb{R}^d$ , namely, the solution of the stochastic differential equation,

$$d\Phi(t) = u(\Phi(t))dt + M(\Phi(t))dB(t), \quad t \geq 0, \quad \Phi(0) = x, \quad (2)$$

where  $u = (u_1, u_2, \dots, u_d)^T : \mathsf{X} \rightarrow \mathbb{R}^d$  and  $M : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^k$  are Lipschitz, and  $B = \{B(t) : t \geq 0\}$  is  $k$ -dimensional standard Brownian motion. If the function  $h : \mathsf{X} \rightarrow \mathbb{R}$  is  $C^2$  then we can write [13],

$$\mathcal{D}h(x) = \sum_i u_i(x) \frac{d}{dx_i} h(x) + \frac{1}{2} \sum_{ij} \Sigma_{ij}(x) \frac{d^2}{dx_i dx_j} h(x), \quad x \in \mathsf{X}.$$

The Lyapunov condition considered in this paper is Condition (V3) of [9]: For a function  $V : \mathsf{X} \rightarrow (0, \infty]$  which is finite for at least one  $x \in \mathsf{X}$ , a function  $f : \mathsf{X} \rightarrow [1, \infty)$ , a constant  $b < \infty$ , and a closed, small set  $C \in \mathcal{B}$ ,

$$\mathcal{D}V \leq -\delta f + b\mathbb{I}_C. \quad (V3)$$

It is entirely analogous to its discrete-time counterpart [11], in which the extended generator is replaced by a difference operator  $\mathcal{D} = P - I$ , where  $P$  is the transition kernel of the discrete-time chain and  $I$  is the identity operator.

The lower bound  $f \geq 1$  is imposed in (V3) because this function is used to define two norms: One on measurable functions  $g: \mathbf{X} \rightarrow \mathbb{R}$  via,

$$\|g\|_f := \sup_{x \in \mathbf{X}} \frac{|g(x)|}{f(x)},$$

and a second norm on signed measures  $\mu$  on  $(\mathbf{X}, \mathcal{B})$ :

$$\|\mu\|_f = \sup_{g: |g| \leq f} |\mu(g)|.$$

Our main goal is to establish the ergodicity of  $\Phi$  in terms of this norm: There is an invariant measure  $\pi$  for the semi-group  $\{P^t\}$  satisfying,

$$\lim_{t \rightarrow \infty} \|P^t(x, \cdot) - \pi(\cdot)\|_f = 0. \quad (3)$$

The following result is a partial extension of the  $f$ -Norm Ergodic Theorem of [11] to the continuous time setting.

**Theorem 1.1.** *Suppose that the Markov process  $\Phi$  is  $\psi$ -irreducible and aperiodic, and let  $f \geq 1$  be a function on  $\mathbf{X}$ . Then the following conditions are equivalent:*

(i) *The semi-group admits an invariant probability measure  $\pi$  satisfying:*

$$\pi(f) := \int \pi(dx) f(x) < \infty.$$

(ii) *There exists a closed, small set  $C \in \mathcal{B}$  such that,*

$$\sup_{x \in C} \mathbb{E}_x \left[ \int_0^{\tau_C(1)} f(\Phi(t)) dt \right] < \infty, \quad (4)$$

where  $\tau_C(1) := \inf\{t \geq 1 : \Phi(t) \in C\}$  and  $\mathbb{E}_x$  denotes the expectation operator under  $X_0 = x$ .

(iii) *There exists a closed, small set  $C$  and an extended-valued non-negative function  $V$  satisfying  $V(x_0) < \infty$  for some  $x_0 \in \mathbf{X}$ , such that Condition (V3) holds.*

Moreover, if (iii) holds then there exists a constant  $b_f$  such that,

$$\mathbb{E}_x \left[ \int_0^{\tau_C(1)} f(\Phi(t)) dt \right] \leq b_f (V(x) + 1), \quad x \in \mathbf{X} \quad (5)$$

where  $V$  and  $C$  satisfy the conditions of (iii). The set  $S_V = \{x : V(x) < \infty\}$  is absorbing ( $P^t(x, S_V) = 1$  for each  $x \in S_V$  and all  $t \geq 0$ ), and also full ( $\pi(S_V) = 1$ ).

*Proof.* Theorem 1.2 (b) of [10] gives the equivalence of (i) and (ii). Theorem 4.3 of [10] gives the implication (iii)  $\Rightarrow$  (ii), along with the bound (5).

Conversely, if (ii) holds then we can define,

$$V(x) = \int_0^\infty \mathbb{E}_x \left[ f(\Phi(t)) \exp \left( - \int_0^t \mathbb{I}\{\Phi(s) \in C\} ds \right) \right] dt. \quad (6)$$

We show in Proposition 2.2 that this is a solution to (V3) and that it is uniformly bounded on  $C$ .  $\square$

The function  $V$  in (6) has the following interpretation. Let  $\tilde{T}$  denote an exponential random variable that is independent of  $\Phi$ , and denote,

$$\tilde{\tau}_C = \min \left\{ t : \int_0^t \mathbb{I}\{\Phi(s) \in C\} ds = \tilde{T} \right\}.$$

We then have,

$$V(x) = \mathbb{E}_x \left[ \int_0^{\tilde{\tau}_C} f(\Phi(t)) dt \right], \quad (7)$$

where now the expectation is over both  $\Phi$  and  $\tilde{T}$ . Consequently, this construction is similar to the converse theorems found in [11] for discrete-time models.

Theorem 1.1 is almost identical to the *f*-Norm Ergodic Theorem of [11], except that it leaves out the implications to ergodicity of the process. This brings us to two open problems: Under the conditions of Theorem 1.1:

Q1 *Can we conclude that (3) holds for any initial condition  $x \in S_V$ ?*

Q2 *Assume in addition that  $\pi(V) < \infty$ . Can we conclude that there exists a finite constant  $B_f$  such that, for all  $x \in S_V$ ,*

$$\int_0^\infty \|P^t(x, \cdot) - \pi\|_f dt \leq B_f(V(x) + 1). \quad (8)$$

In discrete time, questions Q1 and Q2 are answered in the affirmative by the *f*-Norm Ergodic Theorem of [11], with the integral replaced by a sum in (8).

Q2 is resolved in the affirmative in this paper by an application of the discrete-time counterpart:

**Theorem 1.2.** *Suppose that the Markov process  $\Phi$  is  $\psi$ -irreducible and aperiodic, and that there is a solution to (V3) with  $V$  everywhere finite. Then there is a constant  $B_f^0$  such that for each  $x, y \in \mathbf{X}$ ,*

$$\int_0^\infty \|P^t(x, \cdot) - P^t(y, \cdot)\|_f dt \leq B_f^0(V(x) + V(y) + 1) \quad (9)$$

*If in addition  $\pi(V) < \infty$ , then (8) also holds for some constant  $B_f$  and all  $x$ .*

Although the full resolution of Q1 remains open, in Section 3 we discuss how (3) can be established under additional conditions on the process  $\Phi$ .

We begin, in the following section, with the proof of the implication (ii)  $\Rightarrow$  (iii), which is based on theory of generalized resolvents and *f*-regularity [8]. Following this result, it is shown in Proposition 2.3 that *f*-regularity of the process is equivalent to  $f_\Delta$ -regularity for the sampled process, where  $\Delta$  is the sampling interval, and,

$$f_\Delta(x) = \int_0^\Delta \mathbb{E}_x[f(\Phi(t))] dt, \quad x \in \mathbf{X}. \quad (10)$$

This is the basis of the proof of Theorem 1.2 that is contained in Section 3.

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## 2 *f*-Regularity

Following [8], we denote for each  $r \geq 0$  and  $B \in \mathcal{B}$ ,

$$G_B(x, f; r) := \mathbb{E}_x \left[ \int_0^{\tau_B(r)} f(\Phi(t)) dt \right], \quad (11)$$

where  $\tau_B(r) = \inf\{t \geq r : \Phi(t) \in B\}$ , and we write  $G_B(x, f) = G_B(x, f; 0)$ . The Markov process is called *f-regular* if there exists  $r_0 > 0$  such that  $G_B(x, f; r_0) < \infty$  for every  $x$  and every  $B \in \mathcal{B}$  satisfying  $\psi(B) > 0$ .

The following result, given here without proof, is a simple consequence of Lemma 4.1 and Prop. 4.3 of [8]:

**Proposition 2.1.** *Suppose that the set  $C$  is closed and small, and that the following self-regularity property holds: There exists  $r_0 > 0$  such that  $\sup_{x \in C} G_C(x, f; r_0) < \infty$ . Then:*

- (i) *There is  $b_C < \infty$  such that  $G_C(x, f; r) < G_C(x, f; r_0) + b_C r$  for each  $x$  and  $r$ .*
- (ii) *For each  $B \in \mathcal{B}$  satisfying  $\psi(B) > 0$ , for each  $r \geq 0$ , and for each  $x \in \mathbf{X}$ ,*

$$G_C(x, f; r) < \infty \Rightarrow G_B(x, f; r) < \infty.$$

Consequently, the process is *f-regular* if  $G_C(x, f; r_0) < \infty$  for each  $x$ .

We next show that the function  $V$  in (6) is finite-valued on  $\{x \in \mathbf{X} : G_C(x, f; r_0) < \infty\}$ . We show that  $V$  is in the domain of the extended generator, and obtain an expression for  $\mathcal{D}V$ .

Consider the generalized resolvent developed in [8, 12]: For a function  $h: \mathbf{X} \rightarrow \mathbb{R}_+$ ,  $A \in \mathcal{B}$ , and  $x \in \mathbf{X}$ , denote,

$$R_h(x, A) = \int_0^\infty \mathbb{E}_x \left[ \mathbb{I}_A(\Phi(t)) \exp \left( - \int_0^t h(\Phi(s)) ds \right) \right] dt.$$

With the usual interpretation of  $P^t$ , or any kernel  $Q(x, dy)$ , as a lineal operator,  $g \mapsto Qg = \int g(y)Q(\cdot, dy)$ , it is shown in [12] that the following resolvent equation holds: For any functions  $g \geq h \geq 0$ ,

$$R_h = R_g + R_g I_{g-h} R_h, \quad (12)$$

where, for any function  $g$ ,  $I_g$  denotes the (operator induced by the) kernel  $I_g(x, dy) = g(x)\delta_x(dy)$ .

When  $h \equiv \alpha$  is constant, we obtain the usual resolvent,

$$R_\alpha := \int_0^\infty e^{-\alpha t} P^t dt, \quad \alpha > 0, \quad (13)$$

In the case  $\alpha = 1$  we write  $R := R_1 = \int_0^\infty e^{-t} P^t dt$ , and call  $R$  “the” resolvent kernel. For any non-negative function  $g: \mathbf{X} \rightarrow \mathbb{R}_+$  for which  $Rg$  is finite valued, the function  $\gamma = Rg$  is in the domain of the extended generator, with,

$$\mathcal{D}\gamma = Rg - g. \quad (14)$$

**Proposition 2.2.** *Suppose that the assumptions of Theorem 1.1 (ii) hold: There is a closed, small set  $C \in \mathcal{B}$  such that,  $\sup_{x \in C} G_C(x, f; r_0) < \infty$  with  $r_0 = 1$ . Then the function  $V$  defined in (7) is finite on the full set  $S_V \subset \mathbf{X}$  and (V3) holds with this function  $V$  and this closed set  $C$ .*

*Proof.* Proposition 4.3 (ii) of [8] implies that the set of  $x$  for which  $G_C(x, f; 1) < \infty$  is a full set. This result combined with Proposition 4.4 (ii) of [8] implies that  $V$  is bounded on  $C$ .

For arbitrary  $x$  we have  $\tilde{\tau}_C > \tau_C = \min\{t \geq 0 : \Phi(t) \in C\}$ . Consequently, by the strong Markov property and the representation (7),

$$\begin{aligned} V(x) &= \mathbb{E}_x \left[ \int_0^{\tau_C} f(\Phi(t)) dt \right] + \mathbb{E}_x \left[ \mathbb{E}_{\Phi(\tau_C)} \left[ \int_0^{\tilde{\tau}_C} f(\Phi(t)) dt \right] \right] \\ &\leq G_C(x, f; 1) + \sup_{x' \in C} V(x'). \end{aligned}$$

Hence  $V(x)$  is finite whenever  $G_C(x, f; 1)$  is finite.

To establish (V3), first observe that the function  $V$  in (7) can be expressed,

$$V = R_h f, \quad \text{with } h = \mathbb{I}_C.$$

Taking  $g \equiv 1$ , the resolvent equation gives,

$$R_h = R + RI_{1-h}R_h = R[I + I_{C^c}R_h],$$

where, for any set  $B$  and kernel  $Q$ ,  $I_B Q$  denotes the kernel  $\mathbb{I}_B(x)Q(x, dy)$ . Combining the representation of  $V$  above with (14) we obtain,

$$\begin{aligned} V &= R[I + I_{C^c}R_h]f \\ \text{and} \quad \mathcal{D}V &= (R - I)[I + I_{C^c}R_h]f. \end{aligned}$$

The second equation can be decomposed as follows,

$$\mathcal{D}V = D_1 - D_2 - f,$$

with  $D_1 = R[I + I_{C^c}R_h]f = V$  and  $D_2 = I_{C^c}R_h f = I_{C^c}V$ . Substitution then gives,

$$\mathcal{D}V = -f + \mathbb{I}_C V.$$

This establishes (V3) with  $b = \sup_{x \in C} V(x)$ . □

The final results in this section concern the  $\Delta$ -skeleton chain. This is the discrete-time Markov chain with transition kernel  $P^\Delta$ , where  $\Delta \geq 1$  is given. It can be realized by sampling the Markov process with sampling interval  $\Delta$ . The sampled process is denoted,

$$X(i) = \Phi(i\Delta), \quad i \geq 0. \quad (15)$$

In prior work, the skeleton chain is used to translate ergodicity results for discrete-time Markov chains to the continuous time setting. For example, Theorem 6.1 of [9] implies that a weak version of the ergodic convergence (3) holds for an  $f$ -regular Markov process:

$$\lim_{t \rightarrow \infty} \|P^t(x, \cdot) - \pi(\cdot)\|_1 = 0. \quad (16)$$

The proof consists of two ingredients: (i) The corresponding ergodicity result holds for the  $\Delta$ -skeleton chain, and (ii) the error  $\|P^t(x, \cdot) - \pi(\cdot)\|_1$  is non-increasing in  $t$ .

In the next section we use a similar approach to address question Q2. The  $f_\Delta$  norm is considered, where the function  $f_\Delta$  is defined in (10). Denote,

$$\sigma_C^\Delta = \min\{i \geq 0 : X(i) \in C\}, \quad \tau_C^\Delta = \min\{i \geq 1 : X(i) \in C\}.$$

The  $\Delta$ -skeleton is called  $f_\Delta$ -regular if,

$$G_B^\Delta(x, f_\Delta) := \mathbb{E}_x \left[ \sum_{i=0}^{\tau_B^\Delta} f_\Delta(X(i)) \right] < \infty,$$

for every  $x \in \mathsf{X}$  and every  $B \in \mathcal{B}$  satisfying  $\psi(B) > 0$ .

**Proposition 2.3.** *If the process  $\Phi$  is  $f$ -regular, then each  $\Delta$ -skeleton is  $f_\Delta$ -regular. Moreover, there is a closed  $f$ -regular set  $C$  such that:*

- (i) *For a finite-valued function  $V_\Delta : \mathsf{X} \rightarrow (0, \infty]$  and a finite constant  $b$ ,*

$$P^\Delta V_\Delta \leq V_\Delta - f_\Delta + b\mathbb{I}_C, \quad (17)$$

*and  $\sup_x |V_\Delta(x) - G_C(x, f)| < \infty$ .*

- (ii) *For every  $x \in \mathsf{X}$  and every  $B \in \mathcal{B}$  satisfying  $\psi(B) > 0$ , there is a constant  $c_B < \infty$  such that,*

$$G_B^\Delta(x, f_\Delta) \leq G_C(x, f) + c_B. \quad (18)$$

*Proof.* It is enough to establish (i). Theorem 14.2.3 of [11] then implies that for every  $B \in \mathcal{B}$  satisfying  $\psi(B) > 0$ , there is a constant  $c_B^\Delta < \infty$  satisfying  $G_B^\Delta(x, f_\Delta) \leq V_\Delta(x) + c_B^\Delta$ .

Let  $C$  denote any closed  $f$ -regular set for the process, satisfying  $\psi(C) > 0$ . For  $V_0(x) = G_C(x, f)$  we obtain a bound similar to (17) through the following steps. First write,

$$P^\Delta V_0(x) = \mathbb{E}_x \left[ \int_\Delta^{\tau_C(\Delta)} f(\Phi(t)) dt \right].$$

The integral can be expressed as a sum,

$$\begin{aligned} \int_{\Delta}^{\tau_C(\Delta)} f(\Phi(t)) dt &= \int_{\Delta}^{\tau_C(\Delta)} f(\Phi(t)) dt \mathbb{I}\{\tau_C \leq \Delta\} \\ &\quad + \int_{\Delta}^{\tau_C} f(\Phi(t)) dt \mathbb{I}\{\tau_C > \Delta\}. \end{aligned}$$

By the strong Markov property,

$$\begin{aligned} \mathbb{E}_x \left[ \mathbb{I}\{\tau_C \leq \Delta\} \int_{\Delta}^{\tau_C(\Delta)} f(\Phi(t)) dt \right] &\leq \mathbb{E}_x \left[ \mathbb{I}\{\tau_C \leq \Delta\} \int_{\tau_C}^{\tau_C(\Delta)} f(\Phi(t)) dt \right] \\ &\leq \mathbb{P}_x\{\tau_C \leq \Delta\} \sup_y G_C(y, f; \Delta). \end{aligned}$$

Consequently,

$$P^{\Delta} V_0(x) \leq \mathbb{E}_x \left[ \int_{\Delta}^{\tau_C} f(\Phi(t)) dt \right] + b_0 s(x) = V_0(x) - f_{\Delta}(x) + b_0 s(x), \quad (19)$$

where  $b_0 = \sup_y G_C(y, f; \Delta) < \infty$ , and  $s(x) = \mathbb{P}_x\{\tau_C \leq \Delta\}$ .

To eliminate the function  $s$  in (19) we establish the following bound: For some  $\varepsilon_0 > 0$  and  $k_0 \geq 1$ ,

$$P^{k_0 \Delta}(x, C) \geq \varepsilon_0 s(x), \quad x \in \mathbb{X}. \quad (20)$$

The proof is again by the strong Markov property:

$$\begin{aligned} P^{k_0 \Delta}(x, C) &\geq \mathbb{E}_x[\mathbb{I}\{\tau_C \leq \Delta\} \mathbb{I}\{\Phi(k_0 \Delta) \in C\}] \\ &= \int_{r=0}^{\Delta} \int_y \mathbb{P}_x\{\tau_C \in dr, \Phi(r) \in dy\} P^{k_0 \Delta - r}(y, C) \\ &\geq \varepsilon(k) s(x), \end{aligned}$$

where  $\varepsilon(k) = \inf\{P^{k_0 \Delta - r}(y, C) : y \in C, 0 \leq r \leq \Delta\}$ . This is strictly positive for sufficiently large  $k$  because (16) holds. This establishes (20).

The Lyapunov function can now be specified as,

$$V_{\Delta}(x) = V_0(x) + b_0 G_C^{\Delta}(x, s),$$

where  $b_0$  is defined in (19). The required bound  $\sup_x |V_{\Delta}(x) - G_C(x, f)| < \infty$  holds because  $V_0(x) = G_C(x, f)$ , and the second term is uniformly bounded:

$$\begin{aligned} G_C^{\Delta}(x, s) &= \mathbb{E}_x \left[ \sum_{i=0}^{\tau_B^{\Delta}} s(X(i)) \right] \\ &\leq \varepsilon_0^{-1} \mathbb{E}_x \left[ \sum_{i=0}^{\tau_B^{\Delta}} P^{k_0 \Delta}(\Phi(i\Delta), C) \right] \\ &= \varepsilon_0^{-1} \mathbb{E}_x \left[ \sum_{i=0}^{\tau_B^{\Delta}} \mathbb{I}\{X(i + k_0) \in C\} \right] \leq \varepsilon_0^{-1} (k_0 + 1). \end{aligned}$$



Consequently, from familiar arguments,

$$\begin{aligned} PV_\Delta(x) - V_\Delta(x) &\leq -f_\Delta(x) + b_0 s(x) \\ &\quad + b_0 \left\{ G_C^\Delta(x, s) - s(x) + \mathbb{I}_C(x) \varepsilon_0^{-1} (k_0 + 1) \right\}. \end{aligned}$$

This establishes (17) with  $b = b_0 \varepsilon_0^{-1} (k_0 + 1)$ .  $\square$

### 3 *f*-Norm Ergodicity

In this section we consider the implications to the ergodicity of the process. We assume that (V3) holds for a finite-valued function  $V : \mathbf{X} \rightarrow (0, \infty)$ , so that the process is *f*-regular.

**Q1. *f*-norm ergodicity.** The ergodicity of  $\Phi$  in terms of the *f*-norm as in (3) has only been established under special conditions. Theorem 5.3 of [10] implies that (3) will hold if *f* is subject to this additional bound: For some  $\beta \geq 0$ ,

$$P^t f \leq \beta e^{\beta t} f, \quad t \geq 0.$$

This holds for example if  $f \equiv 1$  and  $\beta = 1$ .

It is likely that the application of coupling bounds will lead to a more general theory. Under stronger conditions on the process, such a coupling time was obtained in [6], and it was used in [7] to obtain rates of convergence in the law of large numbers. However, to construct the coupling time, it is assumed in this prior work that the semi-group  $\{P^t\}$  admits a density for each  $t$ . No such assumptions are required in the discrete-time setting, so the full answer to Q1 remains open.

**Q2. Proof of Theorem Theorem 1.2.** The copmplete resolution of Q2 is possible by applying Proposition 2.3, which implies that the skeleton chain  $\{X(i) = \Phi(i\Delta) : i \geq 0\}$  is  $f_\Delta$ -regular. The bound (18) is the main ingredient in the proof of Theorem 1.2, but we also require the following relationship between a norm for the process and a norm for the sampled chain.

**Lemma 3.1.** *For any signed measure  $\mu$ ,*

$$\|\mu\|_{f_\Delta} \geq \int_0^\Delta \|\mu P^t\|_f dt,$$

where, for any measure  $\nu$  and kerner  $Q$ ,  $\nu Q$  denotes the measure  $\nu Q(\cdot) = \int \nu(dx) Q(x, \cdot)$ .

*Proof.* We first consider the right-hand side. Consider the signed measure  $\Gamma$  on  $[0, \Delta] \times \mathbf{X}$  defined by:

$$\Gamma(dt, dy) = \mu P^t(dy) dt.$$

Define  $f^\Delta : [0, \Delta] \times \mathbf{X} \rightarrow [1, \infty)$  via  $f(t, y) = f(y)$  for each pair  $t, y$ , and the associated norm,

$$\|\Gamma\|_{f^\Delta} = \sup \iint g(t, y) \Gamma(dt, dy),$$

where the supremum is over all  $g$  satisfying  $|g(t, y)| \leq f^\Delta(t, y)$  for all  $t, y$ . It is shown next that the norm can be expressed,

$$\|\Gamma\|_{f^\Delta} = \int_0^\Delta \|\mu P^t\|_f dt. \quad (21)$$

The Jordan decomposition theorem [2] implies that there is a minimal decomposition,  $\Gamma = \Gamma_+ - \Gamma_-$ , in which the two measures on the right-hand side are non-negative, with disjoint supports denoted  $S_+, S_-$ , respectively. Hence  $|\Gamma| := \Gamma_+ + \Gamma_-$  is a non-negative measure. In this notation the norm is expressed,

$$\begin{aligned} \|\Gamma\|_{f^\Delta} &= \iint f^\Delta(t, y) |\Gamma|(dt, dy) \\ &= \iint f(y) (\mathbb{I}_{S_+}(t, y) - \mathbb{I}_{S_-}(t, y)) \Gamma(dt, dy) \\ &= \int_0^\Delta \left[ \int_{y \in X} f(y) (\mathbb{I}_{S_+}(t, y) - \mathbb{I}_{S_-}(t, y)) \mu P^t(dy) \right] dt. \end{aligned}$$

For each  $t$ , the measure on  $(X, \mathcal{B})$  defined by  $(\mathbb{I}_{S_+}(t, y) - \mathbb{I}_{S_-}(t, y)) \mu P^t(dy)$  is the marginal of  $|\Gamma|$ , and is hence a non-negative measure for a.e.  $t$ . It follows that for such  $t$ ,

$$\int_{y \in X} f(y) (\mathbb{I}_{S_+}(t, y) - \mathbb{I}_{S_-}(t, y)) \mu P^t(dy) = \|\mu P^t\|_f,$$

which gives (21).

Consider next the left-hand side of the inequality in the lemma. Letting  $\mu = \mu_+ - \mu_-$  denote the Jordan decomposition for the signed measure  $\mu$ , and  $|\mu| = \mu_+ + \mu_-$ , we have,

$$\|\mu\|_{f^\Delta} = \int f^\Delta(x) |\mu|(dx) = \int_{t=0}^\Delta \int_{x \in X} |\mu|(dx) P^t(x, dy) f(y).$$

The right-hand side can be expressed as,

$$\int_0^\Delta \int |\mu|(dx) P^t(x, dy) f(y) = \iint f^\Delta(t, y) \Lambda_+(dt, dy) + \iint f^\Delta(t, y) \Lambda_-(dt, dy),$$

where  $\Lambda_\pm(dt, dy) = \mu_\pm P^t(dy) dt$  defines a decomposition:

$$\Gamma = \Lambda_+ - \Lambda_-.$$

It follows that  $\|\mu\|_{f^\Delta} \geq \|\Gamma\|_{f^\Delta}$ , by the minimality of the Jordan decomposition. This bound combined with (21) completes the proof.  $\square$

*Proof of Theorem 1.2.* Theorem 1.1 combined with Proposition 2.3 establishes  $f_\Delta$ -regularity of the skeleton chain under (V3): The skeleton chain satisfies (V3) with Lyapunov function  $V_\Delta$  that satisfies  $\sup_x |V_\Delta(x) - G_C(x, f)| < \infty$ . The bound (5) in Theorem 1.1 implies that  $V_\Delta(x) \leq b_f^\Delta(V(x) + 1)$  for some constant  $b_f^\Delta$  and all  $x$ .

Theorem 14.3.4 of [11] then gives the bound, for some finite constant  $M_f^0 < \infty$ ,

$$\sum_{k=0}^{\infty} \|P^{\Delta k}(x, \cdot) - P^{\Delta k}(y, \cdot)\|_{f_{\Delta}} \leq M_f^0(V(x) + V(y) + 1). \quad (22)$$

Next apply Lemma 3.1 with  $\mu(\cdot) = P^{\Delta k}(x, \cdot) - P^{\Delta k}(y, \cdot)$  to obtain,

$$\|P^{\Delta k}(x, \cdot) - P^{\Delta k}(y, \cdot)\|_{f_{\Delta}} \geq \int_0^{\Delta} \|\mu P^t\|_f dt, \quad (23)$$

and recognize that  $\mu P^t(\cdot) = P^{\Delta k+t}(x, \cdot) - P^{\Delta k+t}(y, \cdot)$ . Substituting the resulting bound into (22) establishes (9).

The proof of (8) is similar: If in addition  $\pi(V) < \infty$ , then Theorem 14.3.5 of [11] gives, for some constant  $M_f < \infty$ ,

$$\sum_{k=0}^{\infty} \|P^{\Delta k}(x, \cdot) - \pi(\cdot)\|_{f_{\Delta}} \leq M_f(V(x) + 1). \quad (24)$$

This combined with (23) completes the proof.  $\square$

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